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A mapping-independent analysis for the spectra of the non-commutative two-dimensional harmonic oscillator

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Abstract We propose a mapping-independent method to analyze the spectra of the non-commutative two-dimensional harmonic oscillator. The path integral formulation is applied and an effective Lagrangian is obtained. The spectra are read off directly and the question of uniqueness is answered.

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Because of the development of string theory [1–3], there has been a lot of interest in non-commutativity in recent years. It is known that the open string's end points are non-commutative in the presence of background NS–NS B-fields. This means that the coordinates along the D-branes' world volume, which open strings are attached to, are non-commutative [4–9]. The fluctuations of branes are described by non-commutative field theories. As a result, there are numerous papers dealing with field theories on non-commutative spaces. It has been proven in [10, 11] that some non-trivial phenomena (such as UV/IR mixing) would occur in perturbative quantum field theories. The non-perturbative aspect of field theories on non-commutative spaces has also been studied extensively after the work of [12] was published.

Non-commutative quantum mechanics (NCQM) has also been studied considering various aspects [13–16]. The eigenvalue problem of the NCQM is one of the focuses. The common way of studying the NCQM is to map the non-commutative space to a commutative one and use the operator form to study these problems [17–20]. For the case that only the coordinates are non-commutative, the map from the non-commutative space to the commutative one is unique. However, if we consider non-commutative phase space, i.e., both the coordinates and the momenta are non-commutative, the map from the non-commutative phase space to the commutative one will not be unique [21]. Different maps will lead to different representations in commutative phase space. The natural question can then be raised of whether all the representations in the commutative phase space have the same spectra, or equivalently, whether non-commutative models have unique spectra. The answer to this question is not very obvious in operator form, since in fact the number of possible representations is infinite.

The path integral formulation [22–24] in NCQM has been discussed in [25–27]. In the present letter, we shall employ the path integral formulation to answer the above question. We shall perform the path integral directly without mapping to ordinary phase space. As a result, the question of whether the spectra are unique can be answered.

The key point of the path integral formulation is the construction of the propagator. Both the eigenvalues and the wave functions can be read off from it. As an example for illustrating our method, we shall study the spectra of the non-commutative two-dimensional harmonic oscillator from the path integral point of view. Although we only analyze a simple model, we hope that this letter may shed some light on analogous problems.

We consider the non-commutative phase space, which is characterized by the following commutative relations:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \qquad [\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j, \qquad [\hat{p}_i, \hat{p}_j] = iB_{ij}, \tag{1}$$

where θ^{ij} and B_{ij} are constant antisymmetric matrices. Since we only concentrate on the two-dimensional non-commutative harmonic oscillator, the antisymmetry matrices can be simply chosen $\theta^{ij} = \theta\epsilon^{ij}$ and $B_{ij} = B\epsilon_{ij}$.

The above commutation relations show that in the non-commutative phase space there are two additional Heisen-

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berg-type uncertainty relations:

$$\Delta x^1 \Delta x^2 \geq \theta, \qquad \Delta p_1 \Delta p_2 \geq B, \tag{2}$$

which mean that the wave functions both in coordinate space, $\psi(\vec{r},t) = \langle \vec{r},t|\psi\rangle$, and in momentum space, $\varphi(\vec{p},t) = \langle \vec{p},t|\varphi\rangle$, lose their exact meaning.

The common way to study quantum mechanics in non-commutative phase space is to map non-commutative phase space (1)) to ordinary phase space, which is described by

$$\big[\hat{X}^i, \hat{X}^j\big] = 0, \qquad \big[\hat{X}^i, \hat{P}_j\big] = i\hbar\delta^i_j, \qquad [\hat{P}_i, \hat{P}_j] = 0, \tag{3}$$

via the following transformation [21]:

$$\hat{x}^i = a^i_j \hat{X}^j + b^{ij}\hat{P}_j, \qquad \hat{p}_j = c^i_j \hat{X}^j + d^{ij}\hat{P}_j, \tag{4}$$

in which $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are $\mathbf{2}\times\mathbf{2}$ transformation matrices. One can easily get the conditions that the transformation matrices should satisfy. These are [21]

$$\mathbf{ab}^T - \mathbf{ba}^T = \theta, \qquad \mathbf{cd}^T - \mathbf{dc}^T = B, \qquad \mathbf{ad}^T - \mathbf{bc}^T = I, \tag{5}$$

where T stands for transposition. Obviously, the number of unknown elements of the transformation matrices is larger than the number of the equations, following (5). So the transformation matrices cannot be determined uniquely by (5). It means that the map from the non-commutative phase space (1) to the standard one (3) is not unique. For the case of the two-dimensional harmonic oscillator, one can get different representations by making different choices. For example, if we choose the diagonality of $\mathbf{a}$ and $\mathbf{c}$ as $a_{ij} = a_{(i)}\delta_{ij}$, $c_{ij} = c_{(i)}\delta_{ij}$ and further require $\mathbf{a}^T\mathbf{b} + \mathbf{d}^T\mathbf{c} = 0$, then we will get an anisotropy representation which is the representation in terms of two one-dimensional commutative anisotropic harmonic oscillators. If we choose the matrices $\mathbf{a}$ and $\mathbf{c}$ to be diagonal but with a single eigenvalue each, $a_{ij} = a\delta_{ij}$, $c_{ij} = c\delta_{ij}$, and further require $b_{ij} = b\epsilon_{ij}$ and $d_{ij} = d\epsilon_{ij}$, then we will obtain a representation which is a commutative isotropic two-dimensional harmonic oscillator with an additional term proportional to the two-dimensional angular momentum [21].

Actually, the possible representations are infinite. And some of these representations may look very different. It is natural to wonder whether all these representations have the same spectra, or equivalently, whether the two-dimensional harmonic oscillator in the non-commutative phase space (1) has a unique spectrum. The answer to this question is not very obvious within the operator formulation, since there are too many possible representations.

We shall employ the path integral formulation to investigate the spectrum of the non-commutative harmonic oscillator. We shall perform the path integral directly in non-commutative phase space without mapping it to the commutative one. So the question of uniqueness can be answered.

The classical Hamiltonian of the two-dimensional harmonic oscillator is (here, for simplicity, we set $m = \omega = 1$)

$$H = \frac{1}{2}(p_i)^2 + \frac{1}{2}\big(x^i\big)^2 \tag{6}$$

where the x^i and p_i satisfy the classical version of (1):

$$\big\{x^i, x^j\big\}_{\mathrm{PB}} = \theta\epsilon^{ij}, \qquad \big\{x^i, p_j\big\}_{\mathrm{PB}} = \delta^i_j,$$
$$\{p_i, p_j\}_{\mathrm{PB}} = B\epsilon_{ij}, \tag{7}$$

in which $\{,\}_{\mathrm{PB}}$ stands for the extended Poisson brackets, defined by

$$\big\{F\big(x^i,p_i\big), G\big(x^i,p_i\big)\big\}_{\mathrm{PB}} = \frac{\partial F}{\partial x^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x^i} + \theta^{ij}\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x^j} + B_{ij}\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j} \tag{8}$$

for arbitrary functions $F(x^i, p_i)$ and $G(x^i, p_i)$.

It is straightforward to check that the following Lagrangian gives the commutative relation (7):

$$L = \frac{1}{2\kappa}\big(p_i\dot{x}^i - x^i\dot{p}_i + \theta\epsilon^{ij}p_i\dot{p}_j + B\epsilon_{ij}x^i\dot{x}^j\big) - H, \tag{9}$$

by using the Faddeev–Jackiw method [28]. Here κ is a dimensionless parameter,

$$\kappa = 1 - \theta B, \tag{10}$$

and H has been given in (6).

The above Lagrangian is ill-defined when κ takes its critical value $\kappa = 0$, i.e., $B = B_c = \frac{1}{\theta}$ at first glance. The only way to resolve this question is to set

$$p_i = B_c\epsilon_{ij}x^j, \tag{11}$$

or equivalently,

$$x^i = -\theta\epsilon^{ij}p_j \tag{12}$$

when κ takes a value zero.

The above relations, (11) or (12), show that the system will reduce effectively to the one-dimensional case when the parameter κ tends to zero. To see this clearly, we substitute (11) into the Hamiltonian (6). We get

$$H = \frac{1 + B_c^2}{2}\big(x_1^2 + x_2^2\big). \tag{13}$$

Notice the extended Poisson brackets (7) between x^i; we can write the eigenvalues of the above Hamiltonian (13) directly. They are

$$E_n = \hbar\left(B_c + \frac{1}{B_c}\right)\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots. \tag{14}$$

This result can also be obtained after we calculate the spectra of the Hamiltonian of (6).

The key point of the path integral formulation is to construct the propagator. It can be defined either in coordinate space or in momentum space in ordinary quantum mechanics. However, in the present case, we cannot define the propagator in the coordinate space by $K(\vec{r}_b, t_b|\vec{r}_a, t_a)$ or in the momentum space by $K(\vec{p}_b, t_b|\vec{p}_a, t_a)$ due to the commutative relations (1). We notice that x^1 commutes with p_2, so we can define the propagator by

$$\begin{aligned} &K\big(x_b^1, p_{2b}, t_b\big|x_a^1, p_{2a}, t_a\big) \\ &\quad\equiv\big\langle x_b^1, p_{2b}, t_b\big|x_a^1, p_{2a}, t_a\big\rangle \\ &\quad=\big\langle x_b^1, p_{2b}\big|\exp\left\{-i\frac{H}{\hbar}T\right\}\big|x_a^1, p_{2a}\big\rangle, \end{aligned} \tag{15}$$

where $T = t_b - t_a$ (or symmetrically, we can also choose x^2 and p_1).

The propagator (15) can be written as [25]

$$\begin{aligned} &K\big(x_b^1, p_{2b}, t_b\big|x_a^1, p_{2a}, t_a\big) \\ &\quad=\prod_{k=1,2}\int \mathcal{D}x^k\mathcal{D}p_k\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b}\mathrm{d}t\,L\right\}, \end{aligned} \tag{16}$$

where L is the Lagrangian (9).

We denote by $|n\rangle$ the eigenstates of the Hamiltonian (6) with the corresponding eigenvalues E_n:

$$H|n\rangle = E_n|n\rangle; \tag{17}$$

they satisfy

$$\sum_n |n\rangle\langle n| = 1. \tag{18}$$

Sandwiching (18) into (15), we are led to

$$\begin{aligned} &K\big(x_b^1, p_{2b}, t_b\big|x_a^1, p_{2a}, t_a\big) \\ &\quad=\sum_n\exp\left\{-\mathrm{i}\frac{E_n}{\hbar}T\right\}\varphi_n^*\big(x_b^1, p_{2b}\big)\varphi_n\big(x_a^1, p_{2a}\big), \end{aligned} \tag{19}$$

where $\varphi_n(x^1, p_2) = \langle x^1, p_2|n\rangle$ are the eigenfunctions expressed in terms of the variables x^1 and p_2. It means that if we calculate the propagator (16), then we can read both the eigenvalues and the eigenfunctions from it.

Since the wave functions depend only on x^1 and p_2, we must get rid of the variables x^2 and p_1. Notice that the integrand (16) is an exponential of a quadratic form in the variables x^2 and p_1, and we can integrate them directly. The result is

$$\begin{aligned} &K\big(x_b^1, p_{2b}, t_b\big|x_a^1, p_{2a}, t_a\big) \\ &\quad= N\int \mathcal{D}x^1\mathcal{D}p_2\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b}\mathrm{d}t\,L_{\mathrm{eff}}\right\}, \end{aligned} \tag{20}$$

where N is a numerical factor (which is irrelevant, since we only care for the eigenvalues) and L_{eff} is the Lagrangian expressed in terms of the variables x^1 and p_2. We have

$$\begin{aligned} L_{\mathrm{eff}} =& \frac{1}{2\kappa^2}\big[\big(1+B^2\big)\dot{x}_1^2 + 2(\theta+B)\dot{x}_1\dot{p}_2 + \big(1+\theta^2\big)\dot{p}_2^2\big] \\ &- \frac{1}{2}\big(x_1^2 + p_2^2\big). \end{aligned} \tag{21}$$

The propagator (20) can be calculated in the standard way [22–24], i.e., we can write $x^1 = x_{\mathrm{cl}}^1 + \delta x^1$ and $p_2 = p_{2\mathrm{cl}} + \delta p_2$ (where x_{cl}^1 and $p_{2\mathrm{cl}}$ are the solutions of the classical equations of motion, which are obtained by minimizing the action (21)) and then substitute them into the propagator (20). The propagator can be written $K(x_b^1, p_{2b}, t_b|x_a^1, p_{2a}, t_a) = NF(t_b - t_a)\exp\{\frac{\mathrm{i}}{\hbar}S_{\mathrm{eff}}^{\mathrm{cl}}\}$, in which $F(t_b - t_a)$ is the quantum fluctuation and $S_{\mathrm{eff}}^{\mathrm{cl}} = \int_{t_a}^{t_b}\mathrm{d}t\,L_{\mathrm{eff}}(x_{\mathrm{cl}}^1, \dot{x}_{\mathrm{cl}}^1, p_{2\mathrm{cl}}, \dot{p}_{2\mathrm{cl}})$ is the action along the classical trajectory. However, since there is a mixed term, $\dot{x}^1\dot{p}_2$, in the Lagrangian (21), the process of the direct calculation will be very tedious.

We can avoid this tedious calculation and simplify the process by eliminating the mixed term. This can be achieved by an orthogonal transformation. We denote $\vec{\mathbf{X}} = (x^1, p_2)^T$ and $\vec{\mathbf{Y}} = (y^1, y^2)^T$. The orthogonal transformation is $\vec{\mathbf{X}} = (x^1, p_2)^T = \mathcal{O}\vec{\mathbf{Y}} = \mathcal{O}(y^1, y^2)^T$. Explicitly,

$$\begin{pmatrix} x^1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{pmatrix}\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, \tag{22}$$

in which $\mathcal{O}_{ij}$ $(i, j = 1, 2)$ are the elements of the orthogonal matrix. The expressions for them are

$$\begin{aligned} \mathcal{O}_{11} &= -\frac{\mathcal{B}}{\sqrt{\mathcal{B}^2 + 4(\mathcal{A}-\lambda_1)^2}}, \\ \mathcal{O}_{12} &= \frac{\mathcal{B}}{\sqrt{\mathcal{B}^2 + 4(\mathcal{A}-\lambda_2)^2}}, \\ \mathcal{O}_{21} &= \frac{2(\mathcal{A}-\lambda_1)}{\sqrt{\mathcal{B}^2 + 4(\mathcal{A}-\lambda_1)^2}}, \\ \mathcal{O}_{22} &= -\frac{2(\mathcal{A}-\lambda_2)}{\sqrt{\mathcal{B}^2 + 4(\mathcal{A}-\lambda_2)^2}}, \end{aligned} \tag{23}$$

in which

$$\mathcal{A} = \frac{1+B^2}{2}, \qquad \mathcal{B} = \theta + B, \qquad \mathcal{C} = \frac{1+\theta^2}{2} \tag{24}$$

and

$$\lambda_1 = \frac{1}{2}\big[\mathcal{A} + \mathcal{C} + \sqrt{\mathcal{B}^2 + (\mathcal{A}-\mathcal{C})^2}\,\big],$$

$$\lambda_2 = \frac{1}{2}\big[\mathcal{A}+\mathcal{C}-\sqrt{\mathcal{B}^2+(\mathcal{A}-\mathcal{C})^2}\,\big]. \tag{25}$$

The elements of the matrix $\mathcal{O}$ satisfy the following relations:

$$\mathcal{O}_{11}=\mathcal{O}_{22}, \qquad \mathcal{O}_{12}=-\mathcal{O}_{21}. \tag{26}$$

The effective Lagrangian (21) can be written in terms of the new variables y^1 and y^2 as follows:

$$L_{\rm eff}=\frac{1}{2}M_k\dot{y}_k^2-\frac{1}{2}M_k\Omega_k^2y_k^2, \quad k=1,2, \tag{27}$$

where

$$M_k=\frac{2\lambda_k}{\kappa^2} \tag{28}$$

and

$$\Omega_k^2=\frac{\kappa^2}{2\lambda_k}. \tag{29}$$

Obviously, the above Lagrangian, (27), describes two independent orthogonal one-dimensional harmonic oscillators with masses M_i and frequencies Ω_i, respectively. Substituting (24) into (29), we are led to

$$\begin{aligned}\Omega_1&=\frac{1}{2}\big[\sqrt{4+(\theta-B)^2}+(\theta+B)\big],\\ \Omega_2&=\frac{1}{2}\big[\sqrt{4+(\theta-B)^2}-(\theta+B)\big],\end{aligned} \tag{30}$$

for $\kappa>0$. We have

$$\begin{aligned}\Omega_1&=\frac{1}{2}\big[\sqrt{4+(\theta-B)^2}+(\theta+B)\big],\\ \Omega_2&=-\frac{1}{2}\big[\sqrt{4+(\theta-B)^2}-(\theta+B)\big],\end{aligned} \tag{31}$$

for $\kappa<0$.

The propagator can be calculated in terms of the new variables y^1 and y^2 by using the standard method. It is [22–24]

$$\begin{aligned}&K(\vec{Y}_b,t_b|\vec{Y}_a,t_a)\\ &\quad\equiv\langle\vec{Y}_b,t_b|\vec{Y}_a,t_a\rangle=NF(t_b-t_a)\exp\left\{\frac{\rm i}{\hbar}S_{\rm eff}^{\rm cl}\right\}\\ &\quad\sim\sqrt{\prod_{k=1}^{2}\frac{M_k\Omega_k}{2\pi i\hbar\sin\Omega_kT}}\\ &\qquad\times\exp\left\{\sum_{k=1}^{2}\frac{{\rm i}M_k\Omega_k}{2\hbar\sin\Omega_kT}\big[\big(y_{kb}^2+y_{ka}^2\big)\right.\\ &\qquad\left.\times\cos\Omega_kT-2y_{ka}y_{kb}\big]\right\}.\end{aligned} \tag{32}$$

The eigenvalues can be read off from the above propagator directly. They are [22–24]

$$\begin{aligned}&E_{n_1,n_2}=\hbar\Omega_1\left(n_1+\frac{1}{2}\right)+\hbar\Omega_2\left(n_2+\frac{1}{2}\right),\\ &\quad n_1,n_2=0,1,2,\ldots.\end{aligned} \tag{33}$$

The eigenfunctions corresponding to the eigenvalues (33) can also be read off from the propagator (32). They are [22–24]

$$\begin{aligned}\phi_{n_1,n_2}(y_1,y_2)&=\prod_{k=1}^{2}\phi_{n_k}(y_k)\\ &\sim\prod_{k=1}^{2}\exp\left[-\frac{1}{2}\alpha_ky_k\right]H_{n_k}(\alpha_ky_k),\end{aligned} \tag{34}$$

in which the H_n are the Hermite polynomials and the α_k are dimensionless parameters defined by

$$\alpha_k=\sqrt{\frac{M_k\Omega_k}{\hbar}}. \tag{35}$$

The eigenvalues (33) have been obtained in the previous work by mapping non-commutative phase space to a commutative one [21]. However, since the map is not unique, the problem of uniqueness is not very easy to answer within the operator form. Here, we get our results in a mapping-independent way. In fact, it shows that the two-dimensional non-commutative harmonic oscillator has the unique spectra (33), (30) and (31).

The result (14) can also be obtained from (33) by taking the limit $\kappa\to0$. Notice that

$$\lim_{\kappa\to0}\Omega_1=B_c+\frac{1}{B_c}, \qquad \lim_{\kappa\to0}\Omega_2=0 \tag{36}$$

for both $\kappa>0$ and $\kappa<0$. Thus, we get a result which coincides with (14).

To summarize, we study the spectrum of the non-commutative harmonic oscillator by employing the path integral formulation in this letter. Compared with the previous work, we need not map non-commutative phase space to the commutative one; rather, we propose a mapping-independent way to study this problem and get universal results. Although we only analyze a two-dimensional model, the generalization to higher dimensions is straightforward.

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